Triplicity of Quarks and Leptons

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Abstract

Quarks come in three colors and have electric charges in multiples of one-third. There are also three families of quarks and leptons. Whereas the first two properties can be understood in terms of unification symmetries such as $SU(5)$, $SO(10)$, or E_6, why there should only be three families remains a mystery. I propose how all three properties involving the number three are connected in a fivefold application of the gauge symmetry $SU(3)$.

The fundamental building blocks of particle physics are quarks and leptons. The former have electric charges of 2/3 and $-1/3$, and are triplets under the unbroken gauge symmetry $SU(3)_C$ of Quantum Chromodynamics. The latter have electric charges 0 and -1, and do not have $SU(3)_C$ interactions. There are also 3 families of quarks and leptons, each one transforming in the same way under the standard $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group. The ubiquitous occurrence of the number three may be indicative of an underlying symmetry larger than that of the present observed Standard Model. Although each family of quarks and leptons may be considered as components of $\mathbf{5^*} + \mathbf{10}$ under $SU(5)$, or of $\mathbf{16}$ under $SO(10)$, or of $\mathbf{27}$ under E_6, the existence of 3 families remains unexplained in this context.

A strong hint as to what the underlying symmetry could be comes from the maximal subgroup of E_6, i.e. $SU(3)_C \times SU(3)_L \times SU(3)_R$, under which quarks are contained in the representations $(3, 3^*, 1)$ and $(3^*, 1, 3)$, and leptons in $(1, 3, 3^*)$. Here I propose the following extension. Let the gauge symmetry be $SU(3)_C \times SU(3)_L \times SU(3)_M \times SU(3)_R \times SU(3)_F$, where $SU(3)_F$ is the family symmetry and $SU(3)_M$ is the missing link in the lepton sector which allows this scheme to work. The gauge symmetry $SU(3)_F$ is assumed to be broken at or above the unification scale already to its non-Abelian discrete subgroup $\Delta(12)$ [1]. This group is the same as A_4, the symmetry group of the even permutation of four objects. It is also the symmetry group of the regular tetrahedron, one of five perfect geometric solids known to the ancient Greeks and identified by Plato with the element "fire" [2]. There are four irreducible representations of A_4: $\underline{1}$, $\underline{1}'$, $\underline{1}''$, and $\underline{3}$, with the multiplication rule [3]

$$\underline{3} \times \underline{3} = \underline{1} + \underline{1}' + \underline{1}'' + \underline{3} + \underline{3}. \tag{1}$$

Under

$$\mathcal{G} = SU(3)_C \times SU(3)_L \times SU(3)_M \times SU(3)_R \times A_4, \tag{2}$$

the quark and lepton assignments are assumed to be

$$q \sim (3, 3, 1, 1; \underline{3}), \quad q^c \sim (3^*, 1, 1, 3^*; \underline{3}), \tag{3}$$

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$$l \sim (1, 3^*, 3^*, 1; \underline{3}), \quad l^c \sim (1, 1, 3, 3; \underline{3}), \tag{4}$$

with their electric charges given by

$$Q = I_{3L} + \frac{Y_L}{2} + I_{3R} + \frac{Y_R}{2} + I_{3M} - \frac{Y_M}{2}. \tag{5}$$

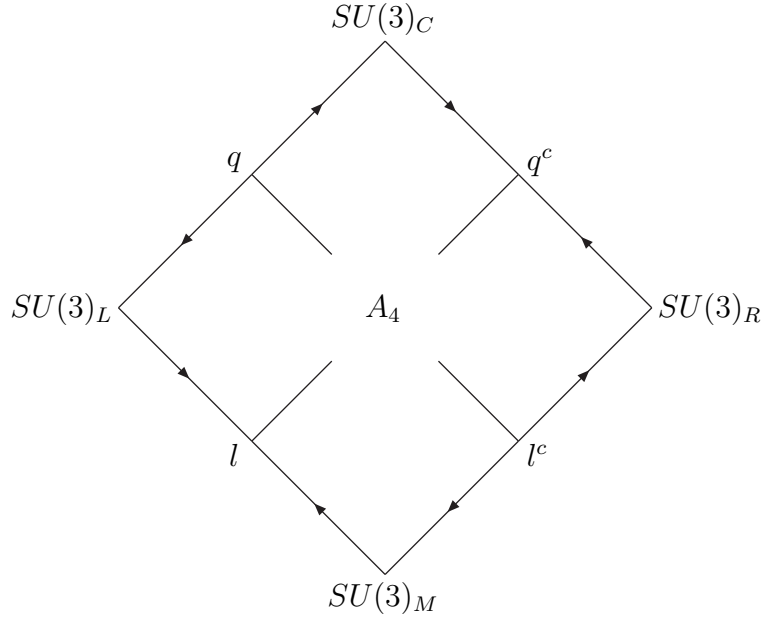A good visual summary of this scheme is Fig. 1.

Figure 1: Pictorial representation of three families of quarks and leptons.

Given the fermionic content of Eqs. (3) and (4), this theory is free of anomalies because each 3 is matched by a 3* of the same multiplicity. There are also three interchange symmetries among the four $SU(3)$ groups, i.e.

$$C \leftrightarrow M, \quad L \leftrightarrow R \; : \; q \leftrightarrow l^c, \quad q^c \leftrightarrow l; \tag{6}$$

$$C \leftrightarrow L, \quad M \leftrightarrow R \; : \; q \leftrightarrow q, \quad l^c \leftrightarrow l^c, \quad l \leftrightarrow q^c; \tag{7}$$

$$C \leftrightarrow R, \quad M \leftrightarrow L \; : \; l \leftrightarrow l, \quad q^c \leftrightarrow q^c, \quad q \leftrightarrow l^c. \tag{8}$$

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These may be used to enforce the equality of the corresponding four gauge couplings, i.e. g_C, g_L, g_M, g_R, at the unification scale.

Using the convention of previous models based on $[SU(3)]^3$ [4], $[SU(3)]^4$ [5], and $[SU(3)]^6$ [6], where the rows have $I_3 = (1/2, -1/2, 0)$ and the columns have $I_3 = (-1/2, 1/2, 0)$, the four matter multiplets are denoted as follows.

$$SU(3)_C \times SU(3)_L \quad : \quad q = \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix}, \tag{9}$$

$$SU(3)_R \times SU(3)_C \quad : \quad q^c = \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix}, \tag{10}$$

$$SU(3)_L \times SU(3)_M \quad : \quad l = \begin{pmatrix} N_1 & E_2^c & \nu \\ E_1 & N_2 & e \\ N_3 & E_3^c & S_1 \end{pmatrix}, \tag{11}$$

$$SU(3)_M \times SU(3)_R \quad : \quad l^c = \begin{pmatrix} N_4 & E_5^c & N_6 \\ E_4 & N_5 & E_6 \\ \nu^c & e^c & S_2 \end{pmatrix}. \tag{12}$$

In the above, the known quarks (u, d, u^c, d^c) and leptons (ν, e, ν^c, e^c) have their usual charges, i.e. $(2/3, -1/3, -2/3, 1/3)$ and $(0, -1, 0, 1)$ respectively. The exotic fields (h, h^c, N, E, E^c, S) have charges $(-1/3, 1/3, 0, -1, 1, 0)$ as given by Eq. (5). As shown below, the choice of $\underline{3}$ of A_4 allows them all to be superheavy, whereas the more obvious choice of 3 and 3* of $SU(3)$ would not. Details on how the representations of A_4 are embedded in those of $SU(3)$ are given in the Appendix.

This model is now extended to include supersymmetry, and its low-energy particle content assumed to be that of the Minimal Supersymmetric Standard Model (MSSM). Assuming the equality of all four gauge couplings at the unification scale, the value of $\sin^2 \theta_W$ from the contributions of q, l^c, l, and q^c is given by

$$\sin^2 \theta_W = \frac{\sum I_{3L}^2}{\sum Q^2} = \frac{\frac{3}{2} + \frac{3}{2} + 0 + 0}{2 + 4 + 4 + 2} = \frac{3}{12} = \frac{1}{4}, \tag{13}$$

which is not equal to the desired values of 3/8 for gauge-coupling unification [7]. However, if only the interchange symmetry of Eq. (7) is used, and

$$g_M^2 = g_R^2 = 2g_L^2 = 2g_C^2 \tag{14}$$

is assumed, then

$$\sin^2 \theta_W = \frac{\frac{3}{2} + \frac{3}{2} + 0 + 0}{2 + 3 + 2 + 1} = \frac{3}{8} \tag{15}$$

as desired. The origin of Eq. (14) is possibly the result of $SU(3)_C$ being the diagonal subgroup of $SU(3)_{CL} \times SU(3)_{CR}$, and $SU(3)_L$ that of $SU(3)_{qL} \times SU(3)_{lL}$, in which case $g_C^{-2} = g_{CL}^{-2} + g_{CR}^{-2}$ and $g_L^{-2} = g_{qL}^{-2} + g_{lL}^{-2}$, so that Eq. (14) is naturally obtained.

The quark and lepton multiplets of Eqs. (3) and (4) are now supermultiplets together with two additional Higgs superfields

$$\Sigma \sim (1, 3, 1, 3^*; \underline{1}), \quad \Sigma^c \sim (1, 3^*, 1, 3; \underline{1}). \tag{16}$$

From Eq. (5), it is obvious that both have the charge assignments of l and l^c of Eqs. (11) and (12). From the invariant terms $\Sigma\Sigma^c$ and $ll^c\Sigma$, it is also clear that the vacuum expectation values of the scalar fields $\langle \tilde{\Sigma}_{33} \rangle$, $\langle \tilde{\Sigma}_{33}^c \rangle$, $\langle \tilde{S}_1 \rangle$, and $\langle \tilde{S}_2 \rangle$ may all be naturally of order the unification scale. The symmetry breaking pattern is given by

$$\langle \tilde{\Sigma}_{33} \rangle, \langle \tilde{\Sigma}_{33}^c \rangle \neq 0 \;\; \Rightarrow \;\; SU(3)_L \times SU(3)_R \to SU(2)_L \times SU(2)_R \times U(1)_{(Y_L+Y_R)/2}, \tag{17}$$

$$\langle \tilde{S}_1 \rangle \neq 0 \;\; \Rightarrow \;\; SU(3)_L \times SU(3)_M \to SU(2)_L \times SU(2)_M \times U(1)_{(Y_L-Y_M)/2}, \tag{18}$$

$$\langle \tilde{S}_2 \rangle \neq 0 \;\; \Rightarrow \;\; SU(3)_M \times SU(3)_R \to SU(2)_M \times SU(2)_R \times U(1)_{(Y_R-Y_M)/2}, \tag{19}$$

resulting in

$$SU(3)_L \times SU(3)_M \times SU(3)_R \to SU(2)_L \times SU(2)_M \times SU(2)_R \times U(1)_{(Y_L+Y_R-Y_M)/2}. \tag{20}$$

The fields which become superheavy at this stage are all the components of Σ and Σ^c, as well as h, h^c from the invariant $q^c q \Sigma^c$ term and $N_3, E_3^c, S_1, N_6, E_6, S_2$ from the invariant

$ll^c\Sigma$ term. To obtain the MSSM, the fields N_1, E_1, E_2^c, N_2 from l and N_4, E_4, E_5^c, N_5 from l^c must also become heavy. Because of the assignments of Eq. (4), the terms lll and $l^c l^c l^c$ are invariant from the product of three bitriplets under two $SU(3)$'s. Thus $N_1 N_2 - E_1 E_2^c$ couples to S_1, and $N_4 N_5 - E_4 E_5^c$ couples to S_2, leaving only the particles of the MSSM (plus ν^c) without any mass. This is possible because both the symmetric and antisymmetric products of three $\underline{3}$'s in A_4 are singlets [3]. If 3 and 3^* of $SU(3)$ are used instead, only the antisymmetric product is a singlet and that always leaves one zero eigenvalue in a 3×3 mass matrix.

The next task is to obtain the breaking

$$SU(2)_M \times SU(2)_R \times U(1)_{(Y_L+Y_R-Y_M)/2} \rightarrow U(1)_{Y/2}. \tag{21}$$

Consider first $\langle \tilde{N}_5 \rangle \neq 0$. This breaks $SU(2)_M \times SU(2)_R$ to $U(1)_{I_{3M}+I_{3R}}$, but does not break $U(1)_{(Y_L+Y_R-Y_M)/2}$. At the same time, $\nu^c N_6 - N_4 S_2$ couples to N_5, so it appears at first sight that ν^c is now massive, but since $N_3 N_6 + E_3^c E_6 + S_1 S_2$ couples to Σ_{33}, a linear combination of ν^c and N_3 will remain massless. Neutrinos remain Dirac fermions in this case. To obtain the desired breaking of Eq. (21), $\langle \tilde{N}_6 \rangle \neq 0$ is also needed. The 9×9 mass matrix spanning $\nu^c, N_3, N_4, N_5, N_6, S_1, S_2, \Sigma$, and Σ^c must now be considered, and a careful analysis shows that ν^c gets a Majorana mass of order

$$m_{\nu^c} \sim \frac{\langle \tilde{N}_5 \rangle^2 \langle \tilde{N}_6 \rangle^2}{M^3}, \tag{22}$$

where M is the unification scale. Note that this does not happen if either $\langle \tilde{N}_5 \rangle = 0$ or $\langle \tilde{N}_6 \rangle = 0$. The origin of this Majorana mass is the 4×4 mass submatrix spanning S_1, S_2, Σ, and Σ^c, due to the $S_1 S_2 \Sigma$ and $\Sigma\Sigma^c$ terms, i.e.

$$\mathcal{M}_{S\Sigma} = \begin{pmatrix} 0 & a & b & 0 \\ a & 0 & c & 0 \\ b & c & 0 & d \\ 0 & 0 & d & 0 \end{pmatrix}, \tag{23}$$

which is not of the Dirac form. Suppose $\langle \tilde{N}_5 \rangle \sim \langle \tilde{N}_6 \rangle \sim 10^{-2} M$, then $m_{\nu^c} \sim 10^{-8} M$, and for $M \sim 10^{16}$ GeV, $m_{\nu^c} \sim 10^8$ GeV which is very suitable for leptogenesis [8, 9]. Note that the presence of $SU(3)_M$ in $\mathcal{G}$ of Eq. (2) is necessary for obtaining this result. Note also that in the well-known $SU(3)^3$ model with only triplets, it is impossible to get a small Majorana neutrino mass. In contrast, the quartic (instead of the canonical quadratic) seesaw formula of Eq. (22) is automatic here in this model.

The last stage of symmetry breaking is

$$SU(2)_L \times U(1)_{Y/2} \to U(1)_Q \tag{24}$$

for which the Higgs superfields

$$\Phi \sim (1, 3, 1, 3^*; \underline{3}), \quad \Phi^c \sim (1, 3^*, 1, 3; \underline{3}) \tag{25}$$

are used, assuming nonzero values of $\langle \tilde{\Phi}_{11} \rangle$, $\langle \tilde{\Phi}_{22} \rangle$, $\langle \tilde{\Phi}^c_{11} \rangle$, $\langle \tilde{\Phi}^c_{22} \rangle$, which allow all leptons and quarks to become massive. The invariant products $l l^c \Phi$ and $q^c q \Phi^c$ have both symmetric and antisymmetric terms under $\underline{3} \times \underline{3} \times \underline{3} \to \underline{1}$ in A_4, resulting in a 3×3 mass matrix for both u and d quarks of the form

$$\mathcal{M}_q = \begin{pmatrix} 0 & a & rc \\ ra & 0 & b \\ c & rb & 0 \end{pmatrix}. \tag{26}$$

Note that if $\mathcal{M}_q$ were antisymmetric, i.e. $r = -1$, then its eigenvalues would be zero and $\pm(|a|^2 + |b|^2 + |c|^2)^{-1/2}$, which is of course unrealistic. On the other hand, if $r = 0$, then its eigenvalues are simply a, b, and c. Now Suppose Φ^c breaks along just one direction in family space, then the ratio $\langle \tilde{\Phi}^c_{11} \rangle / \langle \tilde{\Phi}^c_{22} \rangle$ is the same for each family, which means that $\mathcal{M}_u$ and $\mathcal{M}_d$ are proportional to each other. They will thus undergo the same diagonalization (even if $r \neq 0$) and there will be no mixing. This is exactly the situation in another model of grand unification considered previously [10] and the solution is to generate both the shifts in mass and the mixing matrix from the soft breaking of supersymmetry. Note that $\Phi \Phi^c$,

$\Phi\Phi\Phi$, $\Phi^c\Phi^c\Phi^c$, $\Phi\Phi\Sigma$, and $\Phi^c\Phi^c\Sigma^c$ are all allowed invariant terms under $\mathcal{G}$. This means that there is no theoretical understanding of why the two Higgs doublets of the MSSM are light. This problem is common to all models of grand unification and does not seem to have any special solution in the present context.

Consider now the lepton mass matrices. They come from $ll^c\langle\tilde{\Phi}\rangle$, except that there are also large Majorana mass terms for ν^c already discussed. The canonical seesaw mechanism [11, 12] applies in the usual way and very small Majorana neutrino masses are obtained. Of course, important corrections from soft supersymmetry breaking are applicable in both quark [10] and lepton [13] sectors.

To summarize, it has been poroposed that the fundamental gauge theory of particle interactions is made up at the unification scale of four pairwise interchangeable $SU(3)$ factors, i.e. $SU(3)_C \times SU(3)_L \times SU(3)_M \times SU(3)_R$ plus one additional $SU(3)_F$ which determines the number of families. Quark and lepton families belong to the $\underline{3}$ representation of the discrete non-Abelian subgroup A_4 of $SU(3)_F$. This structure has quark-lepton symmetry at high energies and yet allows naturally for the appearance of the MSSM at low energies. It also allows neutrinos to obtain their small masses through the canonical seesaw mechanism. Soft supersymmetry breaking terms which also break A_4 are necessary, although the details cannot be uniquely determined. In other words, whereas the particle content of this model is the same as that of the MSSM, its soft supersymmetry breaking sector has to be very different and that is something which experiments in the near future can confirm or disprove.

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Appendix. The well-known low-dimensional representations of $SU(3)$ transform under A_4 as follows:

$$3,\ 3^* \sim \underline{3}, \tag{27}$$

$$6,\ 6^* \sim \underline{1} + \underline{1}' + \underline{1}'' + \underline{3}, \tag{28}$$

$$8 \sim \underline{1}' + \underline{1}'' + \underline{3} + \underline{3}, \tag{29}$$

$$10,\ 10^* \sim \underline{1} + \underline{3} + \underline{3} + \underline{3}. \tag{30}$$

Graphically, the $\underline{3}$ of A_4 may be represented by a triangle. There are one, two, and three such (dashed) triangles in the 6, 8, and 10 representations of $SU(3)$ respectively as depicted below.
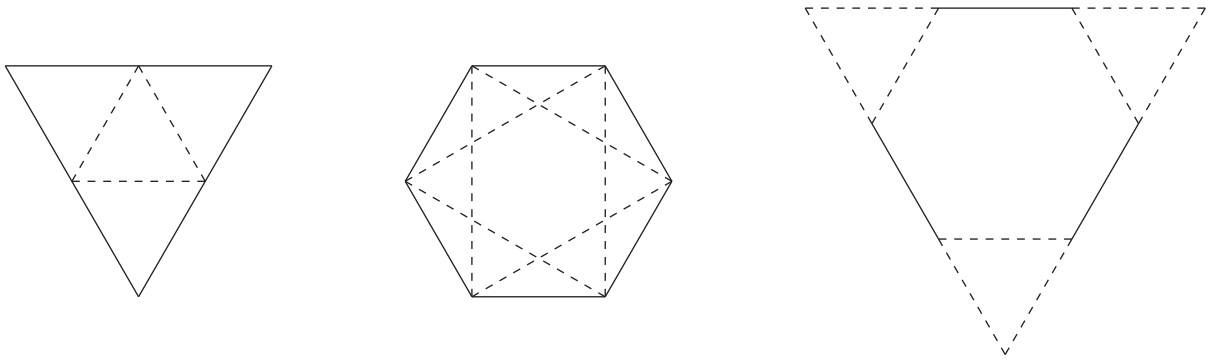

Figure 2: Pictorial representations of the $\underline{3}$ of A_4 in $SU(3)$.